

Peculiarities of squaring method applied to construct solutions of the Dirac, Majorana, and Weyl equations

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O.V. Veko (Kalinkovichi Gymnazium, Belarus, vekoolga@mail.ru)

V.M. Red'kov (Institute of Physics, NAS of Belarus, redkov@dragon.bas-net.by)

Abstract

It is shown that the known method to solve the Dirac equation by means of the squaring method, when relying on the scalar function of the form $\Phi = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \sin(kz + \alpha)$ leads to a 4-dimensional space of the Dirac solutions. It is shown that so constructed basis is equivalent to the space of the Dirac states relied on the use of quantum number $k_1, k_2, \pm k$ and helicity operator; linear transformations relating these two spaces are found. Application of the squaring method substantially depends on the choice of representation for the Dirac matrices, some features of this are considered. Peculiarities of applying the squaring method in Majorana representation are investigated as well. The constructed bases are relevant to describe the Casimir effect for Dirac and Weyl fields in the domain restricted by two planes.

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1 Introduction

In connection with the Casimir effect [1] for the Dirac field in the domain restricted by two planes, of special interest are solutions of the Dirac equation, which have vanishing the third projection of the conserved current J^z on the plane boundaries. Such solutions are reachable when considering 4-dimensional space

$$\{\Psi\} = \{\Psi_{k_1, k_2, k_3=k, \sigma} \otimes \Psi_{k_1, k_2, k_3=-k, \sigma}\}, \quad (1.1)$$

that is the basis of four solutions with opposite signs of the third projection of momentum $+k_3$ and $-k_3$; σ is referred to polarization of the states.

It is shown that the known method to solve the Dirac equation through squaring method [2] (elaboration of such a method to the case of electromagnetic field see in [3, 4]), when relying on the scalar function of the following form

$$\Phi = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \sin(kz + \alpha) \implies \{\Psi\} = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4\} \quad (1.2)$$

leads to a 4-dimensional space of the Dirac solutions. It is shown that so constructed basis (1.2) is equivalent to the space of the Dirac states (1.1); linear transformations relating these two spaces are found. Different values of parameter α in (1.2) determine only different bases in the same linear space (1.2).

Application of the squaring method substantially depends on the choice of representation for the Dirac matrices, some features of this circumstance are considered.

Peculiarities of applying the squaring method in Majorana representation are investigated. It is shown that constructed on the base of scalar functions $\cos(\epsilon t - \vec{k}\vec{x})$ and $-i \sin(\epsilon t - \vec{k}\vec{x})$, two 4-dimensional sets of real and imaginary solutions of the Majorana wave equation cannot be related by any linear transformation.

General conditions for vanishing third projection of the current J^z at the boundaries of the domain between two parallel planes are formulated: firstly, on the basis of plane spinor waves (1.1); and secondly, on the squared basis (1.2). In both cases, these conditions reduce to a linear homogeneous algebraic systems which leads to a 4-th order algebraic equation, the roots of which are e^{2ika} , where a is a half-distance between the planes, and k stands for the third projection of the Dirac particle momentum. Each solution of this equation, with represent a complex number of the unit length, provides us with a certain rule for quantization of the third projection of k . Explicit forms of these algebraic equations are different, however their roots must be the same.

Conditions for vanishing the current J^z for Weyl neutrino field on the boundaries of the domain between two planes are examined, the problem reduce to a 2-nd order algebraic equation.

Covariantization of the conditions for vanishing J^z is performed.

2 Squaring method

Let us start with a special solution of the Klein–Fock–Gordon equation with one formal change in $\Phi = e^{-i\epsilon t + \vec{k}\vec{x}}$: namely, we will change the factor $e^{+ik_3 z}$ by the real-valued factor

$$\sin(kz + \gamma) \equiv \sin \varphi ; \quad (2.1)$$

for brevity we use notation $k = k_3$. Thus, we start with

$$(i\gamma^a \partial_a - M)(i\gamma^a \partial_a + M) = (-\partial_t^2 + \partial_j \partial_j - M^2) ,$$

$$\Phi = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \sin \varphi , \epsilon^2 - k_1^2 - k_2^2 - k^2 - M^2 = 0 . \quad (2.2)$$

The 4×4 - matrix of the four columns-solutions of the Dirac equation is constructed in accordance with the following rule

$$\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\} = (i\gamma^a \partial_a + i\gamma^j \partial_j + M)\Phi . \quad (2.3)$$

In spinor basis, at the choice (2.2), the matrix (2.3) will be

$$\begin{aligned} \Psi_1 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} M \sin \varphi \\ 0 \\ (\epsilon \sin \varphi + ik \cos \varphi) \\ -(k_1 + ik_2) \sin \varphi \end{vmatrix}, \quad \Psi_2 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} 0 \\ M \sin \varphi \\ -(k_1 - ik_2) \sin \varphi \\ (\epsilon \sin \varphi - ik \cos \varphi) \end{vmatrix}, \\ \Psi_3 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} (\epsilon \sin \varphi - ik \cos \varphi) \\ (k_1 + ik_2) \sin \varphi \\ M \sin \varphi \\ 0 \end{vmatrix}, \quad \Psi_4 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} (k_1 - ik_2) \sin \varphi \\ (\epsilon \sin \varphi + ik \cos \varphi) \\ 0 \\ M \sin \varphi \end{vmatrix}. \end{aligned} \quad (2.4)$$

To decide on the linear dependence of Ψ_j (or not), one should examine the following relation

$$a_1 \Psi_1 + a_2 \Psi_2 + a_3 \Psi_3 + a_4 \Psi_4 = 0 ,$$

so we get the linear homogeneous system with respect to a_1, a_2, a_3, a_4 :

$$\begin{aligned} a_1 M \sin \varphi + a_3 (\epsilon \sin \varphi - ik \cos \varphi) + a_4 (k_1 - ik_2) \sin \varphi &= 0 , \\ a_2 M \sin \varphi + a_3 (k_1 + ik_2) \sin \varphi + a_4 (\epsilon \sin \varphi + ik \cos \varphi) &= 0 , \end{aligned}$$

$$\begin{aligned}
a_1(\epsilon \sin \varphi + ik \cos \varphi) - a_2(k_1 - ik_2) \sin \varphi + a_3 M \sin \varphi &= 0, \\
-a_1(k_1 + ik_2) \sin \varphi + a_2(\epsilon \sin \varphi - ik \cos \varphi) + a_4 M \sin \varphi &= 0.
\end{aligned} \tag{2.5}$$

Its determinant turns to be $\det\{\Psi\} = k^4$. This means that four solutions (2.4) of the Dirac equation (at any α in the $\varphi = kz + \alpha$) are linearly independent: they determine a 4-dimensional space.

Let us specify two possibilities for γ in (2.1):

$$\begin{aligned}
\gamma = 0, \quad \varphi = kz, \quad \sin \varphi = \sin kz, \quad \cos \varphi = \cos kz, \\
\Psi_1 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} M \sin kz \\ 0 \\ (\epsilon \sin kz + ik \cos kz) \\ -(k_1 + ik_2) \sin kz \end{vmatrix}, \quad \Psi_2 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} 0 \\ M \sin kz \\ -(k_1 - ik_2) \sin kz \\ (\epsilon \sin kz - ik \cos kz) \end{vmatrix}, \\
\Psi_3 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} (\epsilon \sin kz - ik \cos kz) \\ (k_1 + ik_2) \sin kz \\ M \sin kz \\ 0 \end{vmatrix}, \quad \Psi_4 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} (k_1 - ik_2) \sin kz \\ (\epsilon \sin kz + ik \cos kz) \\ 0 \\ M \sin kz \end{vmatrix},
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\gamma = \frac{\pi}{2}, \quad \varphi' = kz - \frac{\pi}{2}, \quad \sin \varphi' = -\cos kz, \quad \cos \varphi' = \sin kz, \\
\Psi'_1 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} -M \cos kz \\ 0 \\ (-\epsilon \cos kz + ik \sin kz) \\ (k_1 + ik_2) \cos kz \end{vmatrix}, \quad \Psi'_{(2)} = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} 0 \\ -M \cos kz \\ (k_1 - ik_2) \cos kz \\ (-\epsilon \cos kz - ik \sin kz) \end{vmatrix}, \\
\Psi'_3 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} (-\epsilon \cos kz - ik \sin kz) \\ -(k_1 + ik_2) \cos kz \\ -M \cos kz \\ 0 \end{vmatrix}, \quad \Psi'_4 = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} -(k_1 - ik_2) \cos kz \\ (-\epsilon \cos kz + ik \sin kz) \\ 0 \\ -M \cos kz \end{vmatrix}.
\end{aligned} \tag{2.7}$$

We readily find expressions for the following combinations (the total factor $e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y}$ is omitted):

$$\begin{aligned}
-\Psi'_1 + i\Psi_1 = e^{ikz} \begin{vmatrix} M \\ 0 \\ \epsilon - k \\ -(k_1 + ik_2) \end{vmatrix}, \quad -\Psi'_2 + i\Psi_2 = e^{ikz} \begin{vmatrix} 0 \\ M \\ -(k_1 - ik_2) \\ \epsilon + k \end{vmatrix}, \\
-\Psi'_3 + i\Psi_3 = e^{ikz} \begin{vmatrix} \epsilon + k \\ (k_1 + ik_2) \\ M \\ 0 \end{vmatrix}, \quad -\Psi'_4 + i\Psi_4 = e^{ikz} \begin{vmatrix} (k_1 - ik_2) \\ \epsilon - k \\ 0 \\ M \end{vmatrix}.
\end{aligned} \tag{2.8}$$

Note that solutions (2.8) exactly coincide with those obtained by applying the squaring method when one starts with a scalar function as the ordinary plane wave:

$$U = e^{-i\epsilon t + ik_1 x + ik_2 y + ikz} \begin{vmatrix} M & 0 & (\epsilon + k) & (k_1 - ik_2) \\ 0 & M & (k_1 + ik_2) & (\epsilon - k) \\ (\epsilon - k) & -(k_1 - ik_2) & M & 0 \\ -(k_1 + ik_2) & (\epsilon + k) & 0 & M \end{vmatrix}. \tag{2.9}$$

The rank of the matrix in (2.9) equals to 2. Therefore, among four solutions of the Dirac equations given by (2.9) only two of them are linearly independent. For definiteness, let us chose solutions $U_{(1)}$ and

$U_{(2)}$:

$$U_1 = e^{-i\epsilon t + ik_1 x + ik_2 y + ikz} \begin{vmatrix} M \\ 0 \\ \epsilon - k \\ -(k_1 + ik_2) \end{vmatrix}, \quad U_2 = e^{-i\epsilon t + ik_1 x + ik_2 y + ikz} \begin{vmatrix} 0 \\ M \\ -(k_1 - ik_2) \\ \epsilon + k \end{vmatrix}. \quad (2.10)$$

It is readily checked that remaining solutions U_3, U_4 are expressed through U_1, U_2 as follows

$$U_3 = \frac{1}{M} [(\epsilon + k) U_1 + (k_1 + ik_2) U_2], \quad U_4 = \frac{1}{M} [(k_1 - ik_2) U_1 + (\epsilon - k) U_2]. \quad (2.11)$$

Also, instead of (2.8), one can construct other combinations

$$\begin{aligned} -\Psi'_1 - i\Psi_1 &= e^{-ikz} \begin{vmatrix} M \\ 0 \\ \epsilon + k \\ -(k_1 + ik_2) \end{vmatrix}, \quad -\Psi'_2 - i\Psi_2 = e^{-ikz} \begin{vmatrix} 0 \\ M \\ -(k_1 - ik_2) \\ \epsilon - k \end{vmatrix}, \\ -\Psi'_3 - i\Psi_3 &= e^{-ikz} \begin{vmatrix} \epsilon - k \\ (k_1 + ik_2) \\ M \\ 0 \end{vmatrix}, \quad -\Psi'_4 - i\Psi_4 = e^{-ikz} \begin{vmatrix} (k_1 - ik_2) \\ \epsilon + k \\ 0 \\ M \end{vmatrix}. \end{aligned} \quad (2.12)$$

These coincide with those obtained by applying the squaring method to a scalar function as the ordinary plane wave with the only change k into $-k$:

$$U' = e^{-i\epsilon t + ik_1 x + ik_2 y - ikz} \begin{vmatrix} M & 0 & (\epsilon - k) & (k_1 - ik_2) \\ 0 & M & (k_1 + ik_2) & (\epsilon + k) \\ (\epsilon + k) & -(k_1 - ik_2) & M & 0 \\ -(k_1 + ik_2) & (\epsilon - k) & 0 & M \end{vmatrix}. \quad (2.13)$$

Again, the rank of the matrix in (2.13) equals 2, so only two solutions are independent:

$$U'_1 = e^{-i\epsilon t + ik_1 x + ik_2 y - ikz} \begin{vmatrix} M \\ 0 \\ \epsilon + k \\ -(k_1 + ik_2) \end{vmatrix}, \quad U'_{(2)} = e^{-i\epsilon t + ik_1 x + ik_2 y - ikz} \begin{vmatrix} 0 \\ M \\ -(k_1 - ik_2) \\ \epsilon - k \end{vmatrix}. \quad (2.14)$$

Solutions $U_{(3)}$ and $U_{(4)}$ are constructed through $U_{(1)}, U_{(2)}$ in accordance with the rules

$$U'_{(3)} = \frac{1}{M} [(\epsilon - k) U'_{(1)} + (k_1 + ik_2) U_{(2)}], \quad U'_{(4)} = \frac{1}{M} [(k_1 - ik_2) U'_{(1)} + (\epsilon + k) U'_{(2)}]. \quad (2.15)$$

3. Solutions in the basis of momentum 4-vector and helicity

Solutions of the Dirac equation in Cartesian coordinates can be searched in the form

$$\Psi_{\epsilon, k_1, k_2} = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} e^{ik_3 z} \begin{vmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{vmatrix}. \quad (2.16)$$

With the use of spinor basis for Dirac matrices, one gets the linear system for f_i :

$$\begin{aligned} \epsilon f_3 + k_1 f_4 - ik_2 f_4 + k_3 f_3 - M f_1 &= 0, \\ \epsilon f_4 + k_1 f_3 + ik_2 f_3 - k_3 f_4 - M f_2 &= 0, \\ \epsilon f_1 - k_1 f_2 + ik_2 f_2 - k_3 f_1 - M f_3 &= 0, \\ \epsilon f_2 - k_1 f_1 - ik_2 f_1 + k_3 f_2 - M f_4 &= 0. \end{aligned} \quad (2.17)$$

Let us diagonalize the known helicity operator $\Sigma = \sigma_j p_j$. With the substitution (2.16), from eigenvalue equation $\Sigma \Psi = p \Psi$ we arrive at

$$\begin{aligned} k_1 f_2 - i k_2 f_2 + k_3 f_1 &= p f_1, \\ k_1 f_1 + i k_2 f_1 - k_3 f_2 &= p f_2, \\ k_1 f_4 - i k_2 f_4 + k_3 f_3 &= p f_3, \\ k_1 f_3 + i k_2 f_3 - k_3 f_4 &= p f_4. \end{aligned} \quad (2.18)$$

Considering eqs. (2.17) and (2.18) jointly, we obtain the system (note that $p^2 = \epsilon^2 - M^2$)

$$\begin{aligned} \epsilon f_3 + p f_3 - M f_1 &= 0, \quad \epsilon f_4 + p f_4 - M f_2 = 0, \\ \epsilon f_1 - p f_1 - M f_3 &= 0, \quad \epsilon f_2 - p f_2 - M f_4 = 0. \end{aligned}$$

This system results in two values for p and corresponding restrictions on f_i :

$$f_3 = \frac{\epsilon - p}{M} f_1, \quad f_4 = \frac{\epsilon - p}{M} f_2. \quad (2.19)$$

Allowing for (2.19), from (2.18) one gets the system for f_1 and f_2 :

$$\begin{aligned} (k_3 - p) f_1 + (k_1 - i k_2) f_2 &= 0, \\ (k_3 + p) f_2 - (k_1 + i k_2) f_1 &= 0. \end{aligned} \quad (2.20)$$

Further we obtain

$$f_2 = -\frac{i}{i k_1 + k_2} (k_3 - p) f_1 = \frac{k_1 + i k_2}{k_3 + p} f_1.$$

Thus, two independent solutions (we will mark them by α and β) at the fixed \vec{k} are (further let $f_1 = 1$ and $p = +\sqrt{\epsilon^2 - M^2}$)

$$\begin{aligned} (\alpha), \quad \frac{\epsilon - p}{M} &= \alpha, \quad f_1 = 1, \quad f_2 = \frac{k_1 + i k_2}{k_3 + p} = s, \\ (\beta), \quad \frac{\epsilon + p}{M} &= \beta, \quad f_1 = 1, \quad f_2 = \frac{k_1 + i k_2}{k_3 - p} = t; \end{aligned} \quad (2.21)$$

these may be presented in a shorter form

$$\Psi_{(\alpha)} = e^{-i\epsilon t} e^{i k_1 x} e^{i k_2 y} e^{i k_3 z} \begin{vmatrix} 1 \\ s \\ \alpha \\ \alpha s \end{vmatrix}, \quad \Psi_{(\beta)} = e^{-i\epsilon t} e^{i k_1 x} e^{i k_2 y} e^{i k_3 z} \begin{vmatrix} 1 \\ t \\ \beta \\ \beta t \end{vmatrix}, \quad (2.22)$$

3 Relationships between two bases: those obtained from squaring method and the momentum-helicity solutions

Four types of the different solutions of the Dirac equation can be constructed by the method of separation of variables (we take $k_3 = k$ and $k_3 = -k$)

$$\Phi_1 = \Psi_{(\alpha)}(k) = e^{i k z} \begin{vmatrix} 1 \\ \frac{k_1 + i k_2}{k + p} \\ \alpha \\ \alpha \frac{k_1 + i k_2}{k + p} \end{vmatrix}, \quad \Phi_2 = \Psi_{(\alpha)}(-k) = e^{-i k z} \begin{vmatrix} 1 \\ \frac{k_1 + i k_2}{-k + p} \\ \alpha \\ \alpha \frac{k_1 + i k_2}{-k + p} \end{vmatrix},$$

$$\Phi_3 = \Psi_{(\beta)}(k) = e^{ikz} \begin{vmatrix} 1 \\ \frac{k_1+ik_2}{k-p} \\ \beta \\ \beta \frac{k_1+ik_2}{k-p} \end{vmatrix}, \quad \Phi_4 = \Psi_{(\beta)}(-k) = e^{-ikz} \begin{vmatrix} 1 \\ \frac{k_1+ik_2}{-k-p} \\ \beta \\ \beta \frac{k_1+ik_2}{-k-p} \end{vmatrix}. \quad (3.1)$$

and they must be related to four squaring solutions in Section 2.

By physical reason, we should expect existence of the following linear expansions

$$U_1 = a\Phi_1 + b\Phi_3, \quad U_2 = c\Phi_1 + d\Phi_3. \quad (3.2)$$

Evidently, if a, b, c, d are known, one can derive

$$\begin{aligned} U_{(3)} &= \frac{1}{M}[(\epsilon + k) U_{(1)} + (k_1 + ik_2) U_{(2)}] \\ &= \frac{1}{M}[(\epsilon + k)(a\Phi_1 + b\Phi_3) + (k_1 + ik_2)(c\Phi_1 + d\Phi_3)]; \end{aligned} \quad (3.3)$$

$$\begin{aligned} U_{(4)} &= \frac{1}{M}[(k_1 - ik_2) U_{(1)} + (\epsilon - k) U_{(2)}] \\ &= \frac{1}{M}[(k_1 - ik_2)(a\Phi_1 + b\Phi_3) + (\epsilon - k)(c\Phi_1 + d\Phi_3)]. \end{aligned} \quad (3.4)$$

Analogously, there must exist expansions

$$U'_1 = a'\Phi_2 + b'\Phi_4, \quad U'_2 = c'\Phi_2 + d'\Phi_4, \quad (3.5)$$

with the help of which one can derive

$$U'_{(3)} = \frac{1}{M}[(\epsilon - k)(a'\Phi_2 + b'\Phi_4) + (k_1 + ik_2) U_{(2)}], \quad (3.6)$$

$$U'_{(4)} = \frac{1}{M}[(k_1 - ik_2)(a'\Phi_2 + b'\Phi_4) + (\epsilon + k)(c'\Phi_2 + d'\Phi_4)]. \quad (3.7)$$

In turn, after that, one can derive the next expansions by the formulas

$$\Psi_j = \frac{1}{2i}(U_j - U'_j), \quad \Psi'_j = -\frac{1}{2}(U_j + U'_j). \quad (3.8)$$

Let us consider the first relation in (3.2), $U_1 = a\Phi_1 + b\Phi_3$; explicitly it reads

$$\begin{vmatrix} M \\ 0 \\ \epsilon - k \\ -(k_1 + ik_2) \end{vmatrix} = a \begin{vmatrix} 1 \\ \frac{k_1+ik_2}{k+p} \\ \alpha \\ \alpha \frac{k_1+ik_2}{k+p} \end{vmatrix} + b \begin{vmatrix} 1 \\ \frac{k_1+ik_2}{k-p} \\ \beta \\ \beta \frac{k_1+ik_2}{k-p} \end{vmatrix}; \quad (3.9)$$

from whence it follows

$$M = a + b, \quad 0 = \frac{a}{k+p} + \frac{b}{k-p}, \quad \epsilon - k = a\alpha + b\beta, \quad -1 = \frac{a\alpha}{k+p} + \frac{b\beta}{k-p}. \quad (3.10)$$

From the first and second equations we get

$$a = \frac{M}{2p}(k+p), \quad b = -\frac{M}{2p}(k-p); \quad (3.11)$$

two remaining equations turn to be identities.

Now, let us consider the second equation in (3.2): $U_2 = c\Phi_1 + d\Phi_3$; it reads explicitly

$$\begin{vmatrix} 0 \\ M \\ -(k_1 - ik_2) \\ \epsilon + k \end{vmatrix} = c \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{k+p} \\ \alpha \\ \alpha \frac{k_1 + ik_2}{k+p} \end{vmatrix} + d \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{k-p} \\ \beta \\ \beta \frac{k_1 + ik_2}{k-p} \end{vmatrix}; \quad (3.12)$$

or

$$\begin{aligned} 0 &= c + d, & M &= c \frac{k_1 + ik_2}{k+p} + d \frac{k_1 + ik_2}{k-p}, \\ -(k_1 - ik_2) &= c\alpha + d\beta, & \epsilon + k &= \frac{c\alpha(k_1 + ik_2)}{k+p} + \frac{d\beta(k_1 + ik_2)}{k-p}. \end{aligned}$$

From the first and second equations, it follows

$$d = -c, \quad c = \frac{M}{2p}(k_1 - ik_2); \quad (3.13)$$

two remaining ones are identities. Thus, we have obtained

$$U_1 = \frac{M}{2p} [(p+k)\Phi_1 + (p-k)\Phi_3], \quad U_2 = \frac{M}{2p} [(k_1 - ik_2)\Phi_1 - (k_1 - ik_2)\Phi_3]; \quad (3.14)$$

and further

$$U_3 = \frac{M}{2p} [\frac{p+k}{\alpha}\Phi_1 + \frac{p-k}{\beta}\Phi_3], \quad U_4 = \frac{M}{2p} [\frac{k_1 - ik_2}{\alpha}\Phi_1 - \frac{k_1 - ik_2}{\beta}\Phi_3]; \quad (3.15)$$

the identities $\epsilon - p = M/\beta$, $\epsilon + p = M/\alpha$ should be remembered.

Now, let us consider the equation $U'_1 = a'\Phi_2 + b'\Phi_4$; it reads explicitly as

$$\begin{vmatrix} M \\ 0 \\ \epsilon + k \\ -(k_1 + ik_2) \end{vmatrix} = a' \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{-k+p} \\ \alpha \\ \alpha \frac{k_1 + ik_2}{-k+p} \end{vmatrix} + b' \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{-k-p} \\ \beta \\ \beta \frac{k_1 + ik_2}{-k-p} \end{vmatrix}; \quad (3.16)$$

what differs formally from (3.9) only in the change $k \rightarrow -k$ a presence of primes at variables, so the result can be written at once

$$a' = \frac{M}{2p}(-k+p), \quad b' = -\frac{M}{2p}(-k-p).$$

Consider the equation $U'_2 = c'\Phi_2 + d'\Phi_4$; it reads

$$\begin{vmatrix} 0 \\ M \\ -(k_1 - ik_2) \\ \epsilon - k \end{vmatrix} = c' \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{-k+p} \\ \alpha \\ \alpha \frac{k_1 + ik_2}{-k+p} \end{vmatrix} + d' \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{-k-p} \\ \beta \\ \beta \frac{k_1 + ik_2}{-k-p} \end{vmatrix}; \quad (3.17)$$

which differs from (3.12) only in notation, so its solution looks $d' = -c'$, $c' = \frac{M}{2p}(k_1 - ik_2)$.

Thus, we have derived decompositions

$$\begin{aligned} U'_1 &= \frac{M}{2p} [(p-k)\Phi_2 + (p+k)\Phi_4], & U'_2 &= \frac{M}{2p} [(k_1 - ik_2)\Phi_2 - (k_1 - ik_2)\Phi_4], \\ U'_3 &= \frac{M}{2p} \left(\frac{p-k}{\alpha}\Phi_2 + \frac{p+k}{\beta}\Phi_4 \right), & U'_4 &= \frac{M}{2p} \left(\frac{k_1 - ik_2}{\alpha}\Phi_2 - \frac{k_1 - ik_2}{\beta}\Phi_4 \right). \end{aligned} \quad (3.18)$$

and repeat (3.14), (3.15):

$$\begin{aligned} U_1 &= \frac{M}{2p} [(p+k)\Phi_1 + (p-k)\Phi_3], \quad U_2 = \frac{M}{2p} [(k_1 - ik_2)\Phi_1 - (k_1 - ik_2)\Phi_3], \\ U_3 &= \frac{M}{2p} \left(\frac{p+k}{\alpha}\Phi_1 + \frac{p-k}{\beta}\Phi_3 \right), \quad U_4 = \frac{M}{2p} \left(\frac{k_1 - ik_2}{\alpha}\Phi_1 - \frac{k_1 - ik_2}{\beta}\Phi_3 \right). \end{aligned} \quad (3.19)$$

Next, relying on the formulas

$$\begin{aligned} \Psi_1 &= \frac{1}{2i}(U_1 - U'_1), \quad \Psi'_1 = -\frac{1}{2}(U_1 + U'_1), \quad \Psi_2 = \frac{1}{2i}(U_2 - U'_2), \quad \Psi'_2 = -\frac{1}{2}(U_2 + U'_2), \\ \Psi_3 &= \frac{1}{2i}(U_3 - U'_3), \quad \Psi'_3 = -\frac{1}{2}(U_3 + U'_3), \quad \Psi_4 = \frac{1}{2i}(U_4 - U'_4), \quad \Psi'_4 = -\frac{1}{2}(U_4 + U'_4), \end{aligned} \quad (3.20)$$

we derive decompositions

$$\Psi_l = a_{ln}\Phi_n, \quad \Psi'_l = a'_{ln}\Phi_n,$$

where the involved matrices are given by

$$ia_{ij} = \frac{M}{4p} \begin{vmatrix} (p+k) & -(p-k) & (p-k) & -(p+k) \\ (k_1 - ik_2) & -(k_1 - ik_2) & -(k_1 - ik_2) & (k_1 - ik_2) \\ \alpha^{-1}(p+k) & -\alpha^{-1}(p-k) & \beta^{-1}(p-k) & -\beta^{-1}(p+k) \\ \alpha^{-1}(k_1 - ik_2) & -\alpha^{-1}(k_1 - ik_2) & -\beta^{-1}(k_1 - ik_2) & \beta^{-1}(k_1 - ik_2) \end{vmatrix}, \quad (3.21)$$

$$-a'_{ij} = \frac{M}{4p} \begin{vmatrix} (p+k) & (p-k) & (p-k) & (p+k) \\ (k_1 - ik_2) & (k_1 - ik_2) & -(k_1 - ik_2) & -(k_1 - ik_2) \\ \alpha^{-1}(p+k) & \alpha^{-1}(p-k) & \beta^{-1}(p-k) & \beta^{-1}(p+k) \\ \alpha^{-1}(k_1 - ik_2) & \alpha^{-1}(k_1 - ik_2) & -\beta^{-1}(k_1 - ik_2) & -\beta^{-1}(k_1 - ik_2) \end{vmatrix}. \quad (3.22)$$

From those we can construct a new matrix

$$S_{ij} = -a'_{ij} + ia_{ij} = \frac{M}{2p} \begin{vmatrix} (p+k) & 0 & (p-k) & 0 \\ (k_1 - ik_2) & 0 & -(k_1 - ik_2) & 0 \\ \alpha^{-1}(p+k) & 0 & \beta^{-1}(p-k) & 0 \\ \alpha^{-1}(k_1 - ik_2) & 0 & -\beta^{-1}(k_1 - ik_2) & 0 \end{vmatrix}; \quad (3.23)$$

(it corresponds to the use in the factor e^{ikz} in the initial scalar substitution for Φ) The rank of this matrix equals 2, and it is responsible for transformation

$$U_1, U_2, U_3, U_4 \xleftarrow{S} \Phi_1, \Phi_3.$$

Analogously, we have another combination (it corresponds to the use in the factor e^{-ikz} in the initial scalar substitution for Φ)

$$S'_{ij} = -a'_{ij} - ia_{ij} = \frac{M}{2p} \begin{vmatrix} 0 & (p-k) & 0 & (p+k) \\ 0 & (k_1 - ik_2) & 0 & -(k_1 - ik_2) \\ 0 & \alpha^{-1}(p-k) & 0 & \beta^{-1}(p+k) \\ 0 & \alpha^{-1}(k_1 - ik_2) & 0 & -\beta^{-1}(k_1 - ik_2) \end{vmatrix}; \quad (3.24)$$

the rank of this matrix equals 2, and it corresponds to the transformation

$$U'_1, U'_2, U'_3, U'_4 \xleftarrow{S} \Phi_2, \Phi_4.$$

Because the determinants do not vanish

$$\begin{aligned} \det(a_{ij}) &= +\frac{M\alpha^2 k^2}{ip} (k_1 - ik_2)^2 (\alpha^2 + \beta^2 - 2), \\ \det(a'_{ij}) &= -\frac{Mk^2}{ip} (k_1 - ik_2)^2 (\alpha^2 + \beta^2 - 2), \end{aligned} \quad (3.25)$$

inverse matrices exist and have the form

$$[a_{ij}^{-1}] = \frac{2ip}{kM(\alpha - \beta)} = \begin{vmatrix} \alpha & \frac{\alpha(k-p)}{k_1 - ik_2} & -1 & -\frac{(k-p)}{k_1 - ik_2} \\ \alpha & -\frac{\alpha(k+p)}{k_1 - ik_2} & -1 & \frac{(k+p)}{k_1 - ik_2} \\ \beta & \frac{\beta(k+p)}{k_1 - ik_2} & -1 & -\frac{(k+p)}{k_1 - ik_2} \\ \beta & -\frac{\beta(k-p)}{k_1 - ik_2} & -1 & \frac{(k-p)}{k_1 - ik_2} \end{vmatrix}, \quad (3.26)$$

$$[(a')_{ij}^{-1}] = \frac{2ip}{kM(\alpha - \beta)} = \begin{vmatrix} -\alpha & -\frac{\alpha(k-p)}{k_1 - ik_2} & 1 & \frac{(k-p)}{k_1 - ik_2} \\ \alpha & -\frac{\alpha(k+p)}{k_1 - ik_2} & -1 & \frac{(k+p)}{k_1 - ik_2} \\ -\beta & -\frac{\beta(k+p)}{k_1 - ik_2} & 1 & \frac{(k+p)}{k_1 - ik_2} \\ \beta & -\frac{\beta(k-p)}{k_1 - ik_2} & -1 & \frac{(k-p)}{k_1 - ik_2} \end{vmatrix}. \quad (3.27)$$

Evidently, we have inverse transformations relating Ψ_j and Ψ'_j :

$$\Psi'_j = a'_{jk} \Phi_k = [a'_{jk}(a^{-1})_{kl}] \Psi_k,$$

$$a'_{jk}(a^{-1})_{kl} = \frac{i}{2k(\alpha - \beta)} \begin{vmatrix} (p+k)\alpha & -\frac{(p-k)^2\alpha}{k_1 - ik_2} & -(p-k) & \frac{p^2-k^2}{k_1 - ik_2} \\ (k_1 - ik_2)\alpha & -(k+p)\alpha & (k_1 - ik_2) & -(k+p) \\ \frac{(k+p)\beta}{\alpha} & \frac{(p^2-k^2)\beta}{(k_1 - ik_2)\alpha} & -\frac{p-k}{\beta} & -\frac{(p+k)^2}{(k_1 - ik_2)\beta} \\ \frac{(k_1 - ik_2)\beta}{\alpha} & \frac{(p-k)\beta}{\alpha} & \frac{k_1 - ik_2}{\beta} & \frac{p-k}{\beta} \end{vmatrix}, \quad (3.28)$$

$$\Psi_j = a_{jk} \Phi_k = [a_{jk}(a'^{-1})_{kl}] \Psi'_k,$$

$$[a_{jk}(a'^{-1})_{kl}] = \frac{1}{k(\alpha - \beta)} \begin{vmatrix} -(p+k)\alpha & -\frac{(p-k)^2\alpha}{k_1 - ik_2} & (p-k) & \frac{p^2-k^2}{k_1 - ik_2} \\ (k_1 - ik_2)\alpha & (k+p)\alpha & (k_1 - ik_2) & (k+p) \\ \frac{-(k+p)\beta}{\alpha} & \frac{(p^2-k^2)\beta}{(k_1 - ik_2)\alpha} & \frac{p-k}{\beta} & -\frac{(p+k)^2}{(k_1 - ik_2)\beta} \\ \frac{(k_1 - ik_2)\beta}{\alpha} & -\frac{(p-k)\beta}{\alpha} & \frac{k_1 - ik_2}{\beta} & -\frac{p-k}{\beta} \end{vmatrix}. \quad (3.29)$$

The following conclusion can be given: the choice of an initial phase γ in the function $\sin(kz + \gamma)$ does not influence on the whole structure of the space of solutions – it only determines a specific basis in the same space:

$$\begin{aligned} \sin(kz + \gamma) &= \cos \gamma [\sin kz] - \sin \gamma [-\cos kz], \\ \sin(kz + \gamma) &\implies \Psi_j^\gamma = \cos \gamma \Psi_j - \sin \gamma \Psi'_j. \end{aligned} \quad (3.30)$$

4 Dependence of squared solutions on the choice of Dirac matrices, standard basis

Let us follow two representations for Dirac matrices

$$\gamma^a, \quad \Gamma^a = S\gamma^a S^{-1}.$$

Having applied the squaring method in both bases, we will obtain two sets of solutions:

$$(i\gamma^a \partial_a + M)\Phi = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}, \quad (i\Gamma^a \partial_a + M)\Phi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}. \quad (4.1)$$

These two sets, or two matrices, are linked to each other according to the formula

$$S\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}S^{-1} = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}. \quad (4.2)$$

This relation tells that if at a fixed initial scalar function Φ we have a matrix of squared solutions with rank 2, the corresponding matrix of solutions in any other Dirac basis will have also the rank 2. Analogously, if we have the matrix of solutions of the rank 4 in one Dirac basis, we will have in any other Dirac basis a matrix of solutions of the same rank 4.

For instance, let us specify squared solutions in the commonly used standard representation for Dirac matrices:

$$\gamma^0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \quad \gamma^i = \begin{vmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{vmatrix}.$$

Taking into account the relation

$$(i\gamma^a \partial_a + M) = \begin{vmatrix} i\partial_t + M & 0 & i\partial_3 & i\partial_1 + \partial_2 \\ 0 & i\partial_t + M & i\partial_1 - \partial_2 & -i\partial_3 \\ -i\partial_3 & -i\partial_1 - \partial_2 & -i\partial_t + M & 0 \\ -i\partial_1 + \partial_2 & i\partial_3 & 0 & -i\partial_t + M \end{vmatrix},$$

and choosing

$$\Phi = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \sin \varphi, \quad \sin \varphi = kz + \gamma,$$

we get an explicit form for the matrix of solutions

$$[W] = \begin{vmatrix} (\epsilon + M) \sin \varphi & 0 & +ik \cos \varphi & (-k_1 + ik_2) \sin \varphi \\ 0 & (\epsilon + M) \sin \varphi & (-k_1 - ik_2) \sin \varphi & -ik \cos \varphi \\ -ik \cos \varphi & (k_1 - ik_2) \sin \varphi & (-\epsilon + M) \sin \varphi & 0 \\ (k_1 + ik_2) \sin \varphi & +ik \cos \varphi & 0 & (-\epsilon + M) \sin \varphi \end{vmatrix}. \quad (4.3)$$

Thus, we construct four different solutions of the Dirac equation:

$$\begin{aligned} W_1 &= \begin{vmatrix} (\epsilon + M) \sin \varphi \\ 0 \\ -ik \cos \varphi \\ (k_1 + ik_2) \sin \varphi \end{vmatrix}, & W_2 &= \begin{vmatrix} 0 \\ (\epsilon + M) \sin \varphi \\ (k_1 - ik_2) \sin \varphi \\ +ik \cos \varphi \end{vmatrix}, \\ W_3 &= \begin{vmatrix} +ik \cos \varphi \\ (-k_1 - ik_2) \sin \varphi \\ (-\epsilon + M) \sin \varphi \\ 0 \end{vmatrix}, & W_4 &= \begin{vmatrix} (-k_1 + ik_2) \sin \varphi \\ -ik \cos \varphi \\ 0 \\ (-\epsilon + M) \sin \varphi \end{vmatrix}. \end{aligned} \quad (4.4)$$

Due to statement after (4.2), they are linearly independent.

Two choices of the phase γ lead to two sets of solutions:

$$\varphi = kz, \quad \sin \varphi = \sin kz, \quad \cos \varphi = \cos kz,$$

$$\begin{aligned} W_1 &= \begin{vmatrix} (\epsilon + M) \sin kz \\ 0 \\ -ik \cos kz \\ (k_1 + ik_2) \sin kz \end{vmatrix}, & W_2 &= \begin{vmatrix} 0 \\ (\epsilon + M) \sin kz \\ (k_1 - ik_2) \sin kz \\ +ik \cos kz \end{vmatrix}, \\ W_3 &= \begin{vmatrix} +ik \cos kz \\ (-k_1 - ik_2) \sin kz \\ (-\epsilon + M) \sin kz \\ 0 \end{vmatrix}, & W_4 &= \begin{vmatrix} (-k_1 + ik_2) \sin kz \\ -ik \cos kz \\ 0 \\ (-\epsilon + M) \sin kz \end{vmatrix}; \end{aligned} \quad (4.5)$$

$$\varphi' = kz - \frac{\pi}{2}, \quad \sin \varphi' = -\cos kz, \quad \cos \varphi' = \sin kz,$$

$$\begin{aligned}
W'_1 &= \begin{vmatrix} -(\epsilon + M) \cos kz \\ 0 \\ -ik \sin kz \\ -(k_1 + ik_2) \cos kz \end{vmatrix}, & W'_2 &= \begin{vmatrix} 0 \\ -(\epsilon + M) \cos kz \\ -(k_1 - ik_2) \cos kz \\ +ik \sin kz \end{vmatrix}, \\
W'_3 &= \begin{vmatrix} +ik \sin kz \\ (k_1 + ik_2) \cos kz \\ (\epsilon - M) \cos kz \\ 0 \end{vmatrix}, & W'_4 &= \begin{vmatrix} (k_1 - ik_2) \cos kz \\ -ik \sin kz \\ 0 \\ (\epsilon - M) \cos kz \end{vmatrix}.
\end{aligned} \tag{4.6}$$

From these, one easily constructs linear combinations referring to application of the squaring method for two different choices of a scalar function:

$$\begin{aligned}
e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} [-W'_j + iW_j] &= (i\gamma_{stand}^a \partial_a + M) e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} e^{+ikz}, \\
e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} [-W'_j - iW_j] &= (i\gamma_{stand}^a \partial_a + M) e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} e^{-ikz}.
\end{aligned} \tag{4.7}$$

Evidently, one can perform the analysis like given after (2.9)–(2.15) with some minor technical alterations.

5 Squaring method and Majorana fermion

Special interest for the method has any Majorana basis for Dirac matrices. Let us specify one of them as follows

$$\Psi_M = A \Psi_{spinor}, \quad A = \frac{1 - \gamma^2}{\sqrt{2}}, \quad A^{-1} = \frac{1 + \gamma^2}{\sqrt{2}}, \quad \Gamma_M^a = A \gamma^a A^{-1}; \tag{5.1}$$

explicitly these matrices read

$$\begin{aligned}
\gamma_M^0 = +\gamma^0 \gamma^2 &= \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, & \gamma_M^1 = +\gamma^1 \gamma^2 &= \begin{vmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{vmatrix}, \\
\gamma_M^2 = \gamma^2 &= \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, & \gamma_M^3 = +\gamma^3 \gamma^2 &= \begin{vmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{vmatrix}.
\end{aligned}$$

First, let us construct real and pure imaginary solutions of the Dirac equation, starting from the Dirac solutions with n momentum-helicity quantum numbers. To this end, it is enough to translate the Dirac plane waves (2.22) of the types α, β from spinor basis to Majorana one (5.1), and after that we are to separate real and imaginary parts of these solutions. In this way we will construct wave functions for Majorana particles with different charge parities.

Thus, we start with the plane waves of the form

$$\begin{aligned}
\Psi_{(\alpha)} &= e^{-i\epsilon t} e^{i\vec{k}\vec{x}} \begin{vmatrix} 1 \\ s \\ \alpha \\ \alpha s \end{vmatrix}, & \alpha &= \frac{\epsilon - p}{M}, & s &= \frac{k_1 + ik_2}{k_3 + p}, \\
\Psi_{(\beta)} &= e^{-i\epsilon t} e^{i\vec{k}\vec{x}} \begin{vmatrix} 1 \\ t \\ \beta \\ \beta t \end{vmatrix}, & \beta &= \frac{\epsilon + p}{M}, & t &= \frac{k_1 + ik_2}{k_3 - p}.
\end{aligned} \tag{5.2}$$

With the help of the matrix A (5.1) we translate them to the Majorana representation:

$$\Psi_{M(\alpha)} = \frac{e^{-ikx}}{\sqrt{2}} \begin{vmatrix} (1+i\alpha s) \\ (s-i\alpha) \\ -i(s+i\alpha) \\ i(1-i\alpha s) \end{vmatrix}, \quad \Psi_{M(\beta)} = \frac{e^{-ikx}}{\sqrt{2}} \begin{vmatrix} (1+i\beta t) \\ (t-i\beta) \\ -i(t+i\beta) \\ i(1-i\beta t) \end{vmatrix}, \quad (5.3)$$

$$\Psi_{M(\alpha)}^* = \frac{e^{+ikx}}{\sqrt{2}} \begin{vmatrix} (1-i\alpha s^*) \\ (s^*+i\alpha) \\ i(s^*-i\alpha) \\ -i(1+i\alpha s^*) \end{vmatrix}, \quad \Psi_{M(\beta)}^* = \frac{e^{+ikx}}{\sqrt{2}} \begin{vmatrix} (1-i\beta t^*) \\ (t^*+i\beta) \\ i(t^*-i\beta) \\ -i(1+i\beta t^*) \end{vmatrix}; \quad (5.4)$$

where $kx = \epsilon t - \vec{k} \cdot \vec{x}$. Then, from wave functions (5.3), one separates real and imaginary parts:

$$\begin{aligned} R_{M(\alpha)} &= \frac{1}{2}(\Psi_{M(\alpha)} + \Psi_{M(\alpha)}^*) \\ &= \frac{1}{\sqrt{2}} \left\{ \cos kx \begin{vmatrix} 1+i\alpha(s-s^*)/2 \\ (s+s^*)/2 \\ \alpha-i(s-s^*)/2 \\ \alpha(s+s^*)/2 \end{vmatrix} + \sin kx \begin{vmatrix} \alpha(s+s^*)/2 \\ -i(s-s^*)/2-\alpha \\ -(s+s^*)/2 \\ 1-i\alpha(s-s^*)/2 \end{vmatrix} \right\}, \\ I_{M(\alpha)} &= \frac{1}{2}(\Psi_{M(\alpha)} - \Psi_{M(\alpha)}^*) \\ &= \frac{1}{\sqrt{2}} \left\{ \cos kx \begin{vmatrix} i\alpha(s+s^*)/2 \\ (s-s^*)/2-i\alpha \\ -i(s+s^*)/2 \\ i+\alpha(s-s^*)/2 \end{vmatrix} + \sin kx \begin{vmatrix} -i+\alpha(s-s^*)/2 \\ -i(s+s^*)/2 \\ -(s-s^*)/2-i\alpha \\ -i\alpha(s+s^*)/2 \end{vmatrix} \right\}, \end{aligned}$$

and

$$\begin{aligned} R_{M(\beta)} &= \frac{1}{2}(\Psi_{M(\beta)} + \Psi_{M(\beta)}^*) \\ &= \cos kx \begin{vmatrix} 2+i\beta(t-t^*) \\ t+t^* \\ 2\beta-i(t-t^*) \\ \beta(t+t^*) \end{vmatrix} + \sin kx \begin{vmatrix} \beta(t+t^*) \\ -i(t-t^*)-2\beta \\ -(t+t^*) \\ 2-i\beta(t-t^*) \end{vmatrix}, \\ I_{M(\beta)} &= \frac{1}{2}(\Psi_{M(\beta)} - \Psi_{M(\beta)}^*) \\ &= \cos kx \begin{vmatrix} i\beta(t+t^*) \\ (t-t^*)-2i\beta \\ -i(t+t^*) \\ 2i+\beta(t-t^*) \end{vmatrix} + \sin kx \begin{vmatrix} -2i+\beta(t-t^*) \\ -i(t+t^*) \\ -(t-t^*)-2i\beta \\ -i\beta(t+t^*) \end{vmatrix}. \end{aligned}$$

Now let us detail the squaring method relied in the Majorana basis. Starting with

$$(i\gamma_M^a \partial_a + M) = \begin{vmatrix} \partial_1 + M & \partial_t - \partial_3 & 0 & -\partial_2 \\ -\partial_t - \partial_3 & -\partial_1 + M & \partial_2 & 0 \\ 0 & \partial_2 & \partial_1 + M & -\partial_t - \partial_3 \\ -\partial_2 & 0 & \partial_t - \partial_3 & -\partial_1 + M \end{vmatrix}$$

and taking $\Phi = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} e^{ik_3 z}$, we derive an explicit form for a matrix of four solutions

$$\{\Psi_j\} = e^{-ikx} \begin{vmatrix} ik_1 + M & -i\epsilon - ik_3 & 0 & -ik_2 \\ i\epsilon - ik_3 & -ik_1 + M & ik_2 & 0 \\ 0 & ik_2 & ik_1 + M & i\epsilon - ik_3 \\ -ik_2 & 0 & -i\epsilon - ik_3 & -ik_1 + M \end{vmatrix}. \quad (5.5)$$

This matrix can be decomposed into real and imaginary parts, The real part reads

$$\{R_j\} = \begin{vmatrix} M \cos kx + k_1 \sin kx & -\epsilon \sin kx - k_3 \sin kx & 0 & -k_2 \sin kx \\ \epsilon \sin kx - k_3 \sin kx & M \cos kx - k_1 \sin kx & k_2 \sin kx & 0 \\ 0 & k_2 \sin kx & M \cos kx + k_1 \sin kx & \epsilon \sin kx - k_3 \sin kx \\ -k_2 \sin kx & 0 & -\epsilon \sin kx - k_3 \sin kx & M \cos kx - k_1 \sin kx \end{vmatrix}.$$

Because the determinant of this matrix does not vanish

$$\det\{R_j\} = (M^2 \cos^2 kx - (k_1^2 + k_2^2 - \epsilon^2 + k_3^2) \sin^2 kx)^2 = M^4;$$

its columns represent linearly independent solutions of the Majorana equation.

In similar manner we consider the matrix $\{I_j\}$:

$$[I_j] = i \begin{vmatrix} k_1 \cos kx - M \sin kx & -\epsilon \cos kx - k_3 \cos kx & 0 & -k_2 \cos kx \\ \epsilon \cos kx - k_3 \cos kx & -k_1 \cos kx - M \sin kx & k_2 \cos kx & 0 \\ 0 & k_2 \cos kx & k_1 \cos kx - M \sin kx & \epsilon \cos kx - k_3 \cos kx \\ -k_2 \cos kx & 0 & -\epsilon \cos kx - k_3 \cos kx & -k_1 \cos kx - M \sin kx \end{vmatrix},$$

its determinant is

$$\det\{I_j\} = ((k_1^2 + k_2^2 - \epsilon^2 + k_3^2) \cos^2 kx - M^2 \sin^2 kx)^2 = M^4,$$

therefore its columns give linearly independent four solutions of the Majorana equation (with opposite charge parity).

Let us write down explicit form of four real solutions:

$$\begin{aligned} R_1 &= \begin{vmatrix} M \cos kx + k_1 \sin kx \\ \epsilon \sin kx - k_3 \sin kx \\ 0 \\ -k_2 \sin kx \end{vmatrix}, & R_2 &= \begin{vmatrix} -\epsilon \sin kx - k_3 \sin kx \\ M \cos kx - k_1 \sin kx \\ k_2 \sin kx \\ 0 \end{vmatrix}, \\ R_3 &= \begin{vmatrix} 0 \\ k_2 \sin kx \\ M \cos kx + k_1 \sin kx \\ -\epsilon \sin kx - k_3 \sin kx \end{vmatrix}, & R_4 &= \begin{vmatrix} -k_2 \sin kx \\ 0 \\ \epsilon \sin kx - k_3 \sin kx \\ M \cos kx - k_1 \sin kx \end{vmatrix}, \end{aligned} \quad (5.6)$$

and four imaginary solutions

$$\begin{aligned} I_1 &= i \begin{vmatrix} k_1 \cos kx - M \sin kx \\ \epsilon \cos kx - k_3 \cos kx \\ 0 \\ -k_2 \cos kx \end{vmatrix}, & I_2 &= i \begin{vmatrix} -\epsilon \cos kx - k_3 \cos kx \\ -k_1 \cos kx - M \sin kx \\ k_2 \cos kx \\ 0 \end{vmatrix}, \\ I_3 &= i \begin{vmatrix} 0 \\ k_2 \cos kx \\ k_1 \cos kx - M \sin kx \\ -\epsilon \cos kx - k_3 \cos kx \end{vmatrix}, & I_4 &= i \begin{vmatrix} -k_2 \cos kx \\ 0 \\ \epsilon \cos kx - k_3 \cos kx \\ -k_1 \cos kx - M \sin kx \end{vmatrix}. \end{aligned} \quad (5.7)$$

Now, let us specify squared solutions in Majorana basis, when starting with the real scalar function ($kx = \epsilon t - \vec{k}\vec{x}$):

$$\begin{aligned} \Phi &= \cos kx, & (i\gamma_M^a \partial_a + M) \cos kx &= \{\Phi_j\} \\ &= \begin{vmatrix} k_1 \sin kx + M \cos kx & -\epsilon \sin kx - k_3 \sin kx & 0 & -k_2 \sin kx \\ \epsilon \sin kx - k_3 \sin kx & -k_1 \sin kx + M \cos kx & k_2 \sin kx & 0 \\ 0 & k_2 \sin kx & k_1 \sin kx + M \cos kx & \epsilon \sin kx - k_3 \sin kx \\ -k_2 \sin kx & 0 & -\epsilon \sin kx - k_3 \sin kx & -k_1 \sin kx + M \cos kx \end{vmatrix}; \end{aligned} \quad (5.8)$$

note identity $\{\Phi_j\} = \{R_j\}$ – see (5.6).

Similarly, one derives

$$\begin{aligned} \Phi' &= -i \sin kx, & (i\gamma_M^a \partial_a + M) \sin kx &= \{\Phi'_j\} \\ &= i \begin{vmatrix} k_1 \cos kx - M \sin kx & -\epsilon \cos kx - k_3 \cos kx & 0 & -k_2 \cos kx \\ \epsilon \cos kx - k_3 \cos kx & -k_1 \cos kx - M \sin kx & k_2 \cos kx & 0 \\ 0 & k_2 \cos kx & k_1 \cos kx - M \sin kx & \epsilon \cos kx - k_3 \cos kx \\ -k_2 \cos kx & 0 & -\epsilon \cos kx - k_3 \cos kx & -k_1 \cos kx - M \sin kx \end{vmatrix}; \end{aligned} \quad (5.9)$$

again note identity $\{\Phi'_j\} = \{I_j\}$.

It is readily verified, that these two 4-dimensional spaces, R_j and I_j , cannot be connected by any linear transformation. To this end, one should check the relationships

$$I_{(j)} = S_{(j)k} R_{(k)} \iff I_{(j)l} = S_{(j)k} R_{(k)l}. \quad (5.10)$$

Indeed, from (5.10) we infer that the matrix S depends on coordinates

$$\begin{aligned} I = S R &\implies S = I R^{-1} = \frac{i}{M^2} \\ &\times \begin{vmatrix} M k_1 - F \sin 2kx & -(\epsilon + k_3)^2 \sin 2kx & 0 & -k_2 \sin 2kx \\ -(\epsilon - k_3)^2 \sin 2kx & -(M k_1 + F \sin 2kx) & -k_2 \sin 2kx & 0 \\ 0 & -k_2 \sin 2kx & M k_1 - \frac{1}{2}(M^2 + k_1^2) \sin 2kx & -(\epsilon - k_3)^2 \sin 2kx \\ -k_2 \sin 2kx & 0 & -(\epsilon + k_3)^2 \sin 2kx & -(M k_1 + F \sin 2kx) \end{vmatrix}, \end{aligned}$$

where for brevity we use notation $\frac{1}{2}(M^2 + k_1^2) = F$. It means that these two sets, $\{I_j\}$ and $\{R_j\}$, determine substantially different spaces: from physical standpoint they refer respectively to Majorana fermions with different properties with respect to the charge conjugation.

In the charged Dirac case, at fixed momentum there exist sets of two states different in helicity, whereas in Majorana real case we face sets of four solutions only.

6 Vanishing the current J^z on the boundaries of the domain between two planes

Using Dirac plane waves with fixed polarization states (for brevity the general factor $e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y}$ is omitted):

$$\Psi_1 = \Psi_{\alpha,k} = e^{ikz} \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{k+p} \\ \alpha \\ \alpha \frac{k_1 + ik_2}{k+p} \end{vmatrix}, \quad \Psi_2 = \Psi_{(\beta),k} = e^{ikz} \begin{vmatrix} 1 \\ \frac{k_1 + ik_2}{k-p} \\ \beta \\ \beta \frac{k_1 + ik_2}{k-p} \end{vmatrix}, \quad (6.1)$$

and similar ones with the change $k \implies -k$:

$$\Psi_3 = \Psi_{\alpha,-k} = e^{-ikz} \begin{vmatrix} 1 \\ -\frac{k_1 + ik_2}{k-p} \\ \alpha \\ -\alpha \frac{k_1 + ik_2}{k-p} \end{vmatrix}, \quad \Psi_4 = \Psi_{(\beta),-k} = e^{-ikz} \begin{vmatrix} 1 \\ -\frac{k_1 + ik_2}{k+p} \\ \beta \\ -\beta \frac{k_1 + ik_2}{k+p} \end{vmatrix}, \quad (6.2)$$

let us make a linear combination $\Phi = A_1 \Psi_1 + A_2 \Psi_2 + A_3 \Psi_3 + A_4 \Psi_4$. Four components of this wave function Φ are

$$\Phi_1 = e^{-i\epsilon t} e^{iax} [e^{ikz} (A_1 + A_2) + e^{-ikz} (A_3 + A_4)],$$

$$\begin{aligned}
\Phi_2 &= e^{-i\epsilon t} e^{iax} (k_1 + ik_2) \left[e^{ikz} \left(\frac{A_1}{k+p} + \frac{A_2}{k-p} \right) - e^{-ikz} \left(\frac{A_3}{k-p} + \frac{A_4}{k+p} \right) \right], \\
\Phi_3 &= e^{-i\epsilon t} e^{iax} \left[e^{ikz} (A_1\alpha + A_2\beta) + e^{-ikz} (A_3\alpha + A_4\beta) \right], \\
\Phi_4 &= e^{-i\epsilon t} e^{iax} (k_1 + ik_2) \left[e^{ikz} \left(\frac{A_1\alpha}{k+p} + \frac{A_2\beta}{k-p} \right) - e^{-ikz} \left(\frac{A_3\alpha}{k-p} + \frac{A_4\beta}{k+p} \right) \right].
\end{aligned} \tag{6.3}$$

Note the structure of the current (in spinor basis) is

$$J^z = \Phi^+ \gamma^0 \gamma^3 \Phi = (\Phi_1^* \Phi_1 - \Phi_3^* \Phi_3) - (\Phi_2^* \Phi_2 - \Phi_4^* \Phi_4).$$

The current on the boundaries ($z = -a, +a$) vanishes if the following requirements are fulfilled:

$$\begin{aligned}
\underline{z = -a}, \quad \Phi_3 &= e^{i\rho} \Phi_1, \quad \Phi_4 = e^{i\sigma} \Phi_2 \quad \implies \\
&[e^{-ika} (A_1\alpha + A_2\beta) + e^{ika} (A_3\alpha + A_4\beta)] = e^{i\rho} [e^{-ika} (A_1 + A_2) + e^{ika} (A_3 + A_4)], \\
&\left[e^{-ika} \left(\frac{A_1\alpha}{k+p} + \frac{A_2\beta}{k-p} \right) - e^{ika} \left(\frac{A_3\alpha}{k-p} + \frac{A_4\beta}{k+p} \right) \right] = e^{i\sigma} \left[e^{-ika} \left(\frac{A_1}{k+p} + \frac{A_2}{k-p} \right) - e^{ika} \left(\frac{A_3}{k-p} + \frac{A_4}{k+p} \right) \right]; \\
\underline{z = +a}, \quad \Phi_3 &= e^{i\mu} \Phi_1, \quad \Phi_4 = e^{i\nu} \Phi_2 \quad \implies \\
&[e^{ika} (A_1\alpha + A_2\beta) + e^{-ika} (A_3\alpha + A_4\beta)] = e^{i\mu} [e^{ika} (A_1 + A_2) + e^{-ika} (A_3 + A_4)], \\
&\left[e^{ika} \left(\frac{A_1\alpha}{k+p} + \frac{A_2\beta}{k-p} \right) - e^{-ika} \left(\frac{A_3\alpha}{k-p} + \frac{A_4\beta}{k+p} \right) \right] = e^{i\nu} \left[e^{ika} \left(\frac{A_1}{k+p} + \frac{A_2}{k-p} \right) - e^{-ika} \left(\frac{A_3}{k-p} + \frac{A_4}{k+p} \right) \right].
\end{aligned}$$

Thus, we arrive at the homogeneous linear system with respect to complex variables A_1, A_2, A_3, A_4 . It is convenient to introduce notation $K = e^{2iak}$, then the above system reads as

$$\begin{aligned}
A_1(\alpha - e^{i\rho}) + A_2(\beta - e^{i\rho}) + A_3(\alpha - e^{i\rho})K + A_4(\beta - e^{i\rho})K &= 0, \\
A_1K(\alpha - e^{i\mu}) + A_2K(\beta - e^{i\mu}) + A_3(\alpha - e^{i\mu}) + A_4(\beta - e^{i\mu}) &= 0, \\
A_1(\alpha - e^{i\sigma})(k-p) + A_2(\beta - e^{i\sigma})(k+p) - A_3K(\alpha - e^{i\sigma})(k+p) - A_4K(\beta - e^{i\sigma})(k-p) &= 0, \\
A_1K(\alpha - e^{i\nu})(k-p) + A_2K(\beta - e^{i\nu})(k+p) - A_3(\alpha - e^{i\nu})(k+p) - A_4(\beta - e^{i\nu})(k-p) &= 0.
\end{aligned} \tag{6.4}$$

Let us write down an explicit form of the main matrix for linear system (6.4)

$$\begin{vmatrix}
(\alpha - e^{i\rho}) & (\beta - e^{i\rho}) & (\alpha - e^{i\rho})K & (\beta - e^{i\rho})K \\
(\alpha - e^{i\mu})K & (\beta - e^{i\mu})K & (\alpha - e^{i\mu}) & (\beta - e^{i\mu}) \\
(\alpha - e^{i\sigma})(k+p) & (\beta - e^{i\sigma})(k-p) & -(\alpha - e^{i\sigma})(k-p)K & -(\beta - e^{i\sigma})(k+p)K \\
(\alpha - e^{i\nu})(k+p)K & (\beta - e^{i\nu})(k-p)K & -(\alpha - e^{i\nu})(k-p) & -(\beta - e^{i\nu})(k+p)
\end{vmatrix}. \tag{6.5}$$

Eq. $\det S = 0$ reduces to a 4-th order polynomial with respect to K , from physical point of view we are interested in roots which are complex number $K = e^{2ika}$ of the unit length – see the detailed analysis of its solutions in [5].

7 Vanishing the current on the boundaries, basis produced by the squaring method

Now let us derive explicit form of vanishing the current J^z on the boundaries of the domain, when using the set of solutions obtained within the squaring method. We start with (the general factor $e^{-i\epsilon t}e^{ik_1 x}e^{ik_2 y}$ is omitted)

$$\begin{aligned}\Psi_1 &= \begin{vmatrix} M \sin kz \\ 0 \\ (\epsilon \sin kz + ik \cos kz) \\ -(k_1 + ik_2) \sin kz \end{vmatrix}, & \Psi_2 &= \begin{vmatrix} 0 \\ M \sin kz \\ -(k_1 - ik_2) \sin kz \\ (\epsilon \sin kz - ik \cos kz) \end{vmatrix}, \\ \Psi_3 &= \begin{vmatrix} (\epsilon \sin kz - ik \cos kz) \\ (k_1 + ik_2) \sin kz \\ M \sin kz \\ 0 \end{vmatrix}, & \Psi_4 &= \begin{vmatrix} (k_1 - ik_2) \sin kz \\ (\epsilon \sin kz + ik \cos kz) \\ 0 \\ M \sin kz \end{vmatrix}.\end{aligned}\quad (7.1)$$

four components of the linear combination $\Phi = A_1\Psi_1 + A_2\Psi_2 + A_3\Psi_3 + A_4\Psi_4$ are

$$\begin{aligned}\Phi_1 &= A_1 M \sin kz + A_2 0 + A_3(\epsilon \sin kz - ik \cos kz) + A_4(k_1 - ik_2) \sin kz, \\ \Phi_2 &= A_1 0 + A_2 M \sin kz + A_3(k_1 + ik_2) \sin kz + A_4(\epsilon \sin kz + ik \cos kz), \\ \Phi_3 &= A_1(\epsilon \sin kz + ik \cos kz) - A_2(k_1 - ik_2) \sin kz + A_3 M \sin kz + A_4 0, \\ \Phi_4 &= -A_1(k_1 + ik_2) \sin kz + A_2(\epsilon \sin kz - ik \cos kz) + A_3 0 + A_4 M \sin kz.\end{aligned}\quad (7.2)$$

Remembering on the current structure

$$J^z = \Phi^+ \gamma^0 \gamma^3 \Phi = (\Phi_1^* \Phi_1 - \Phi_3^* \Phi_3) - (\Phi_2^* \Phi_2 - \Phi_4^* \Phi_4).$$

we impose restrictions:

$$\underline{z = -a}, \quad \Phi_3 = e^{i\rho} \Phi_1, \quad \Phi_4 = e^{i\sigma} \Phi_2 \quad \implies$$

$$\begin{aligned}& A_1(\epsilon \sin ak - ik \cos ak) - A_2(k_1 - ik_2) \sin ak + A_3 M \sin ak \\ &= e^{i\rho} [A_1 M \sin ak + A_3(\epsilon \sin ak + ik \cos ak) + A_4(k_1 - ik_2) \sin ak], \\ & \quad -A_1(k_1 + ik_2) \sin ak + A_2(\epsilon \sin ak + ik \cos ak) + A_4 M \sin ak \\ &= e^{i\sigma} \Phi_2 [A_2 M \sin ak + A_3(k_1 + ik_2) \sin ak + A_4(\epsilon \sin ak - ik \cos ak)];\end{aligned}$$

$$\underline{z = +a}, \quad \Phi_3 = e^{i\mu} \Phi_1, \quad \Phi_4 = e^{i\nu} \Phi_2 \quad \implies$$

$$\begin{aligned}& A_1(\epsilon \sin ak + ik \cos ka) - A_2(k_1 - ik_2) \sin ak + A_3 M \sin ak \\ &= e^{i\mu} [A_1 M \sin ak + A_3(\epsilon \sin ak - ik \cos ak) + A_4(k_1 - ik_2) \sin ak], \\ & \quad -A_1(k_1 + ik_2) \sin ak + A_2(\epsilon \sin ak - ik \cos ak) + A_4 M \sin ak \\ &= e^{i\nu} \Phi_2 [A_2 M \sin ak + A_3(k_1 + ik_2) \sin ak + A_4(\epsilon \sin ak + ik \cos ak)].\end{aligned}\quad (7.3)$$

With the use of elementary identities

$$\cos ak = \frac{e^{iak} + e^{-iak}}{2} = \frac{K + K^{-1}}{2}, \quad \sin ak = -i \frac{e^{iak} - e^{-iak}}{2} = -i \frac{K - K^{-1}}{2}.$$

and the shortening notations

$$\epsilon + k = m, \quad \epsilon - k = n, \quad k_1 + ik_2 = f, \quad k_1 - ik_2 = g, \quad mn = fg, \quad g = f^*,$$

$$e^{i\rho} = x, \quad e^{i\mu} = y, \quad e^{i\sigma} = v, \quad e^{i\nu} = w ;$$

we arrive at linear equations

$$\begin{aligned} -A_1[K(m-xM) - (n-xM)] + A_2g(K-1) - A_3[K(M-xn) - (M-xm)] + A_4e^{i\rho}g(K-1) &= 0, \\ -A_1[K(n-yM) - (m-yM)] + A_2g(K-1) - A_3[K(M-ym) - (M-yn)] + A_4yg(K-1) &= 0, \\ A_1f(K-1) - A_2[K(n-vM) - (m-vM)] + A_3vf(K-1) - A_4[K(M-vn) - (M-vm)] &= 0, \\ A_1f(K-1) - A_2[K(m-e^{i\nu}M) - (n-e^{i\nu}M)] + A_3e^{i\nu}f(K-1) - A_4[K(M-e^{i\nu}m) - (M-e^{i\nu}n)] &= 0. \end{aligned}$$

This linear homogeneous system has solutions if its determinant equals to zero:

$$S' = \tag{7.4}$$

$$\begin{vmatrix} -K(m-xM) + n - xM & g(K-1) & -K(M-xn) + M - xm & xg(K-1) \\ -K(n-yM) + (m-yM) & g(K-1) & -K(M-ym) + M - yn & yg(K-1) \\ f(K-1) & -K(n-vM) + m - vM & vf(K-1) & -K(M-vn) + M - vm \\ f(K-1) & -K(m-wM) + n - wM & wf(K-1) & -K(M-wm) + M - wn \end{vmatrix}$$

In the paper [6] it was shown that any particular choice in 4-dimensional space of states for the Dirac particle does not influence substantially the result of solving the equations $\det S = 0$ and $\det S' = 0$: the whole set of the roots is the same.

8 2-component Weyl neutrino

The possible way to specify neutrino wave equation is to consider the Dirac equation in spinor basis

$$\psi(x) = \begin{vmatrix} \xi(x) \\ \eta(x) \end{vmatrix}, \quad \xi(x) = \begin{vmatrix} \xi^1 \\ \xi^2 \end{vmatrix}, \quad \eta(x) = \begin{vmatrix} \eta_1 \\ \eta_2 \end{vmatrix}, \quad \gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix}, \tag{8.1}$$

where $\sigma^a = (I, +\sigma^k)$, $\bar{\sigma}^a = (I, -\sigma^k)$

$$\sigma^0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad \sigma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma^2 = \begin{vmatrix} 0 & -i \\ +i & 0 \end{vmatrix}, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

Un this basis, we have two equations

$$i\sigma^a \partial_a \xi(x) = M\eta(x), \quad i\bar{\sigma}^a \partial_a \eta(x) = M\xi(x), \tag{8.2}$$

setting here $m = 0$ we produce equation for neutrino and anti-neutrino:

$$(\text{neutrino}) \quad i\bar{\sigma}^a \partial_a \eta(x) = 0, \quad (\text{anti-neutrino}) \quad i\sigma^a \partial_a \xi(x) = 0. \tag{8.3}$$

The neutrino equation differently reads

$$(i\partial_t - i\partial_j \sigma^j) \eta = 0, \quad \Sigma \eta = -i\partial_t \eta, \tag{8.4}$$

where $p_j \sigma^j = \Sigma$ stands for helicity operator. For the plane wave

$$\eta = e^{-i\epsilon t} e^{i\vec{k}\vec{x}} \begin{vmatrix} \eta_1 \\ \eta_2 \end{vmatrix} \tag{8.5}$$

from (8.4) it follows that helicity of the neutrino with fixed momentum is automatically negative:

$$\Sigma \eta = -\epsilon \eta, \quad \epsilon = +\sqrt{k_1^2 + k_2^2 + k_3^2}.$$

For those states one finds restriction on the components η_1, η_2 in (8.5):

$$\begin{vmatrix} \epsilon + k_3 & k_1 - ik_2 \\ k_1 + ik_2 & \epsilon - k_3 \end{vmatrix} \begin{vmatrix} \eta_1 \\ \eta_2 \end{vmatrix} = 0 \implies \epsilon^2 = k_1^2 + k_2^2 + k_3^2, \quad \eta_2 = -\frac{k_1 + ik_2}{\epsilon - k_3} \eta_1. \tag{8.6}$$

Let the neutrino plane waves be written in the form (for simplicity, $\eta_1 = 1$; and $k_3 = k$; we will need also solutions with opposite k)

$$\eta = e^{-i\epsilon t} e^{ixk_1} e^{iyk_2} e^{izk} \begin{vmatrix} 1 \\ -\frac{k_1+ik_2}{\epsilon-k} \end{vmatrix}, \quad \eta' = e^{-i\epsilon t} e^{ixk_1} e^{iyk_2} e^{-izk} \begin{vmatrix} 1 \\ -\frac{k_1+ik_2}{\epsilon+k} \end{vmatrix}. \quad (8.7)$$

Summing equations for η and η^+ :

$$\eta^+(\partial_t - \vec{\partial}_j \sigma^j) \eta = 0, \quad \eta^+(\partial_t - \partial_j \sigma^j) \eta = 0$$

one derives expression for the neutrino current

$$\partial_a J^a = 0, \quad J^a = (\eta^+ \eta, -\eta^+ \sigma^j \eta). \quad (8.8)$$

Its components are

$$\begin{aligned} J^t &= \eta_1^* \eta_1 + \eta_2^* \eta_2, & J^z &= -\eta_1^* \eta_1 + \eta_2^* \eta_2, \\ J^1 &= -\eta_1^* \eta_2 - \eta_2^* \eta_1, & J^2 &= i\eta_1^* \eta_2 - i\eta_2^* \eta_1. \end{aligned}$$

In particular, the component j^z is given by

$$j^z = -\left(1 - \frac{k_1^2 + k_2^2}{(\epsilon - k)^2}\right) = \frac{k}{\epsilon - k}. \quad (8.9)$$

Now, let us examine the vanishing of the current J^z on the boundaries of the domain between two planes. The structure of the current indicates the way to reach this:

$$J^z = -\eta_1^* \eta_1 + \eta_2^* \eta_2 \implies \eta_2 = e^{i\gamma} \eta_1. \quad (8.10)$$

To satisfy this requirement, let us introduce a special linear combinations of the plane waves with opposite momentums (general factor is omitted)

$$H = A\eta + B\eta' = \begin{vmatrix} Ae^{izk} + Be^{-izk} \\ -\frac{k_1+ik_2}{\epsilon-k} Ae^{izk} - \frac{k_1+ik_2}{\epsilon+k} Be^{-izk} \end{vmatrix}$$

for which the above current restrictions read

$$\begin{aligned} z = -a : \quad & H_2(-a) = e^{i\rho} H_1(-a), \\ & -\frac{k_1+ik_2}{\epsilon-k} Ae^{-iak} - \frac{k_1+ik_2}{\epsilon+k} Be^{iak} = e^{i\rho} [Ae^{-iak} + Be^{iak}]; \\ z = +a : \quad & H_2(a) = e^{i\sigma} H_1(a), \\ & -\frac{k_1+ik_2}{\epsilon-k} Ae^{iak} - \frac{k_1+ik_2}{\epsilon+k} Be^{-iak} = e^{i\sigma} [Ae^{iak} + Be^{-iak}]. \end{aligned}$$

Let us introduce parameter $e^{2iak} = K$, then we get the linear homogeneous system with respect to A and B :

$$\begin{aligned} A(e^{i\rho} + \frac{k_1+ik_2}{\epsilon-k}) + BK(e^{i\rho} + \frac{k_1+ik_2}{\epsilon+k}) &= 0, \\ AK(e^{i\sigma} + \frac{k_1+ik_2}{\epsilon-k}) + B(e^{i\sigma} + \frac{k_1+ik_2}{\epsilon+k}) &= 0. \end{aligned}$$

With the shortening notations

$$\frac{k_1+ik_2}{\epsilon-k} = f, \quad \frac{k_1+ik_2}{\epsilon+k} = g, \quad fg^* = 1, \quad f^*g = 1;$$

the system reads

$$\begin{aligned} A(e^{i\rho} + f) + BK(e^{i\rho} + g) &= 0, \\ AK(e^{i\sigma} + f) + B(e^{i\sigma} + g) &= 0. \end{aligned} \quad (8.11)$$

Nontrivial solution exists if K^2 satisfies the equation below

$$K^2 = \frac{(e^{i\rho} + f)(e^{i\sigma} + g)}{(e^{i\sigma} + f)(e^{i\rho} + g)}. \quad (8.12)$$

Let us simplify the notation: $e^{i\rho} = x$, $e^{i\sigma} = y$, then

$$K^2 = \frac{(x+f)(y+g)}{(x+g)(y+f)}; \quad (8.13)$$

It is ready proved that the right hand part of (8.13) indeed is a complex number of the unit length. To this end, it suffices to multiply it by the complex conjugate – so we obtain the needed identity

$$\frac{(x+f)(y+g)}{(x+g)(y+f)} \frac{(1/x+f^*)(1/y+g^*)}{(1/x+g^*)(1/y+f^*)} = \frac{(x+f)(y+g)}{(x+g)(y+f)} \frac{(g+x)}{(f+x)} \frac{xf}{xg} \frac{(y+f)}{(y+g)} \frac{yg}{yf} \equiv 1.$$

then eq. (8.13) reads as the quantization rule for k in the form

$$e^{4iak} = \frac{(e^{i\rho} + f)(e^{i\sigma} + g)}{(e^{i\rho} + g)(e^{i\sigma} + f)}. \quad (8.14)$$

This equation permits different solutions. Mostly of them can be found only numerically. For the case $e^{i\rho} = e^{i\sigma}$, we have an analytically solvable equation, $e^{4iak} = 1$.

9 On the representation of $J^z = 0$ in arbitrary basis of the Dirac matrices

In spinor basis, vanishing of the current is achieved if

$$\Psi_3 = e^{i\rho} \Psi_1, \quad \Psi_4 = e^{i\sigma} \Psi_2; \quad (9.1)$$

however these relations are not covariant: they are valid only in this particular basis. This deficiency can be resolved. Let us present (9.1) in the matrix form $\Psi = G \Psi$:

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^{-i\rho} & 0 \\ 0 & 0 & 0 & e^{-i\sigma} \\ e^{i\rho} & 0 & 0 & 0 \\ 0 & e^{i\sigma} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}. \quad (9.2)$$

The matrix G can be decomposed in terms of Dirac matrices in spinor representation – such form will be automatically covariant (valid in all bases)

$$\gamma^a = \begin{pmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \gamma^5 \gamma^a = \begin{pmatrix} 0 & -\bar{\sigma}^a \\ \sigma^a & 0 \end{pmatrix}.$$

Decomposition for G (9.2) should be searched in the form

$$G = (n_0 \gamma^0 + n_3 \gamma^3) + \gamma^5 (m_0 \gamma^0 + m_3 \gamma^3); \quad (9.3)$$

which is equivalent to

$$G = \begin{pmatrix} 0 & n_0 - m_0 - (n_3 - m_3) \sigma^3 \\ n_0 + m_0 + (n_3 + m_3) \sigma^3 & 0 \end{pmatrix}$$

and leads to the system

$$\begin{aligned} (n_0 - m_0) - (n_3 - m_3) &= e^{-i\rho}, \\ (n_0 - m_0) + (n_3 - m_3) &= e^{-i\sigma}, \\ (n_0 + m_0) + (n_3 + m_3) &= e^{+i\rho}, \\ (n_0 + m_0) - (n_3 + m_3) &= e^{+i\sigma}. \end{aligned} \quad (9.4)$$

Its solution is

$$\begin{aligned} n_0 &= \frac{1}{2}(\cos \rho + \cos \sigma) , \quad n_3 = \frac{1}{2}(i \sin \rho - i \sin \sigma) , \\ m_0 &= \frac{1}{2}(i \sin \rho + i \sin \sigma) , \quad m_3 = \frac{1}{2}(\cos \rho - \cos \sigma) . \end{aligned}$$

Correspondingly, the matrix G is presented as follows

$$\begin{aligned} G &= \frac{1}{2}(\cos \rho + \cos \sigma)\gamma^0 + \frac{1}{2}(i \sin \rho - i \sin \sigma)\gamma^3 \\ &+ \gamma^5 \left[\frac{1}{2}(i \sin \rho + i \sin \sigma)\gamma^0 + \frac{1}{2}(\cos \rho - \cos \sigma)\gamma^3 \right] ; \end{aligned}$$

This can be translated to other form

$$\begin{aligned} G &= e^{+i\rho} \frac{1+\gamma^5}{2} \frac{\gamma^0+\gamma^3}{2} + e^{-i\rho} \frac{1-\gamma^5}{2} \frac{\gamma^0-\gamma^3}{2} \\ &+ e^{+i\sigma} \frac{1+\gamma^5}{2} \frac{\gamma^0-\gamma^3}{2} + e^{-i\sigma} \frac{1-\gamma^5}{2} \frac{\gamma^0+\gamma^3}{2} . \end{aligned} \quad (9.5)$$

Condition for vanishing the current in covariant form is given by the relation

$$J^z = 0 \quad \Longleftrightarrow \quad G(\rho, \sigma) \Psi = \Psi . \quad (9.6)$$

Formula (9.5) becomes simpler at $e^{i\sigma} = e^{i\rho}$:

$$G = e^{+i\rho} \frac{1+\gamma^5}{2} \gamma^0 + e^{-i\rho} \frac{1-\gamma^5}{2} \gamma^0 ; \quad (9.7)$$

Similarly, (9.5) is simplified if $e^{i\sigma} = e^{-i\rho}$:

$$G = e^{+i\rho} \frac{\gamma^0+\gamma^3}{2} + e^{-i\rho} \frac{\gamma^0-\gamma^3}{2} . \quad (9.8)$$

10. Conclusion

The accent in the paper was given to consideration of solutions of the Dirac equation which have vanishing the third projection of the conserved current J^z on the boundaries of the domain between two parallel planes. Such solutions are reachable when considering 4-dimensional basis of four solutions – plane waves with opposite signs of the third projection of momentum $+k_3$ and $-k_3$. It is shown that the known method to solve the Dirac equation through squaring method, if based on the scalar function $\Phi = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \sin(kz + \alpha)$, leads to 4-dimensional basis of Dirac solutions. It is shown that so constructed basis is equivalent to the 4-space of Dirac plane wave states; corresponding linear transformation is found. Application of the squaring method in Majorana representation is investigated as well.

General conditions for vanishing third projection of the current J^z at the boundaries of the domain between two parallel planes are formulated: on the 4-dimensional base of plane spinor waves and on the base of the squared basis. In both cases, these conditions reduce to linear homogeneous algebraic systems, equation $\det S = 0$ for which turns to be a 4-th order polynomial, the roots of which are e^{2ika} , where a is a half-distance between the planes. Solutions of this 4-th order polynomial will be considered in a separate paper.

The case of Weyl neutrino field is investigated as well; conditions for vanishing the neutrino current J^z on the two boundaries reduce to 2-nd order polynomial: its solutions are found; all of them are complex numbers with the unit length.

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References

- [1] I. V.M Mostenananenko, N.Ya. Trunov. Casimir effect and its applications. Uspekhi Fizicheskix nauk. – 1988. Vol. 156, no 3. P. 385 – 426.
- [2] N.G. Tokarevskaya, V.M. Red'kov. Squaring and separation of the variables in the Dirac equation, in curvilinear coordinates of the Euclidean space. pages 153–160 in: Covariant methods in theoretical physics, the elementary particle physics and relativity theory. Minsk, 2005.
- [3] V.V. Kisel, Ovsyuk E.M., V.M. Red'kov, H.G. Tokarevskaya. Maxwell equations in matrix form, squaring procedure, separating the variables and structure of electromagnetic solutions // Nonlinear Dynamics and Applications. Vol. 16, P. 144 – 168, Minsk, 2009; Proc. of XVI Annual Seminar NPC - 2009, May 19-22, 2009, Minsk, Belarus.
- [4] V.V. Kisel, E.M. Ovsyuk, V.M. Red'kov, N.G. Tokarevskaya. Maxwell equations in complex form, squaring procedure and separating the variables Ricerche di Matematica. - 2011. - Vol. 60. no 1, P. 1-14.
- [5] O.V. Veko, V.M. Red'kov, A.I. Shelest, S.A. Yushchenko, A.M. Ishkhanyan. General conditions for vanishing the current J^z for a spinor field on boundaries of the domain between two planes. Nonlinear Phenomena in Complex Systems. 2014. Vol. 17. no 2. P. 147 – 168
- [6] O.V. Veko, V.M. Red'kov Covariantization of conditions of vanishing the current J^z for the Dirac field at the boundaries of the domain between two planes. Doklady of the National Academy of Sciences of Belarus. 2014. no 3. P. 44 – 50.